

# New multicritical matrix models and multicritical 2d CDT

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## Abstract

We define multicritical CDT models of 2d quantum gravity and show that they are a special case of multicritical generalized CDT models obtained from the new scaling limit, the so-called “classical” scaling limit, of matrix models. The multicritical behavior agrees with the multicritical behavior of the so-called branched polymers.

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# 1 Introduction

Multicritical matrix models have served as an important tool in the study of 2d quantum gravity coupled to matter. They were introduced by Kazakov for this purpose [1] and the identification of the first multicritical matrix model limit as corresponding to a  $(p, q) = (2, 5)$  minimal rational conformal field theory coupled to 2d Euclidean quantum gravity was due to Staudacher [2].

The concept of multicritical so-called branched polymers (BP) was introduced in an attempt to understand the physics underlying the multicritical matrix models [3]. Ordinary branched polymers were encountered in the theory of random surfaces which aimed at providing a non-perturbative definition of the Polyakov path integral, either on a hyper-cubic lattice [5] or using the formalism of dynamical triangulations (DT) [6]. For these non-perturbatively defined random surface theories the result was that in physical dimensions ( $D \geq 2$ ) the random surfaces degenerate into branched polymers [5, 7]. The same phenomenon was observed in attempts to study higher dimensional Euclidean quantum gravity using DT [8]. Branched polymers seem to be very generic structures which are entropically favored and in statistically inspired models of random geometry one has to make special efforts to avoid them.

The formalism of causal dynamical triangulations (CDT) represents an attempt to eradicate the dominance of BP. It has been partly successful as a model of higher dimensional quantum gravity [9]. The two-dimensional CDT model [10] is special in the sense that it is exactly solvable and in the sense that it can be understood as a specific limit of ordinary Euclidean quantum gravity where baby universes have been integrated out [11]. It is somewhat amusing that while one of the purposes of the CDT model was to avoid BP, the two-dimensional model is entirely described by BP, as first noticed in [12]. We can use the BP description of two-dimensional CDT in a constructive way to formulate multicritical CDT models. In the following we will do that, and we will show that these multicritical CDT models are special cases of more general multicritical matrix models. The scaling limit taken for these matrix models, the so-called classical scaling limit, is different from the conventional scaling limit referred to above. This new scaling limit was first introduced in a matrix model with the purpose of obtaining CDT from a matrix model [13]. We present here the generalization to multicritical behavior.

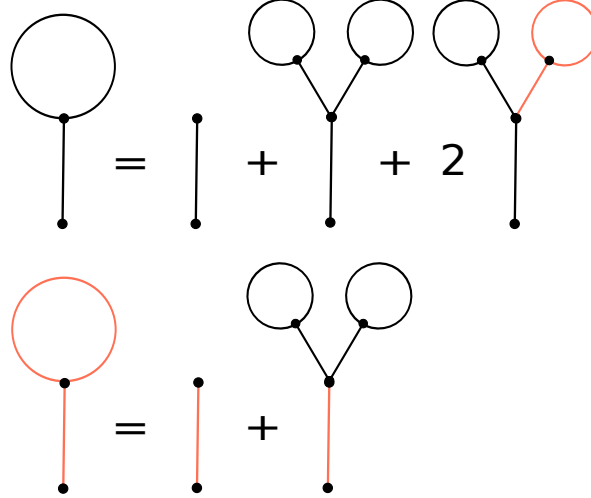


Figure 1: The graphical representation of eqs. (7). The black colored graphs represent the class of BPs where the link touching the root contains no dimer. The red-colored graphs represent the class of BPs where a dimer touches the root. It is assumed that only one link is incident on the root vertex. This restriction can easily be lifted, but is imposed as it leads to slightly simpler algebra.

## 2 Multicritical branched polymers and multicritical CDT

### 2.1 Multicritical branched polymers

One can define a statistical ensemble of BP by the partition function

$$Z(\mu) = \sum_{BP} \prod_i v(i) \prod_l e^{-\mu}. \quad (1)$$

The sum is over graphs which are rooted connected planar trees. The root is a distinguished vertex. For simplicity we assume there is only one link incident on the root. This assumption has no consequences for the critical behavior of the ensemble of BP. To each vertex  $i$  is assigned a weight  $v_i$  which we usually take to depend only on the order of the vertex. To each link  $l$  is assigned a fugacity  $e^{-\mu}$ .

The graphical selfconsistent equation which determines  $Z(\mu)$  is shown as the lower part of Fig. 1, with the colors ignored, in the special case where  $v_1 = v_3 = 1$ , all other  $v_i$  being zero.

It leads to the following equation

$$e^{\mu} = \frac{1 + v_2 Z + v_3 Z^2 \dots}{Z} := \frac{f(Z)}{Z} := F(Z). \quad (2)$$

We assume by simple rescaling that  $v_1 = 1$ . The generic BP is obtained if the weights  $v_i$  are non-negative<sup>1</sup>. In this case  $F(Z)$  has an minimum  $Z_0$  where  $F'(Z_0) = 0$  and  $F''(Z_0) > 0$ . Thus we have the non-analytic behavior

$$Z(\mu) - Z(\mu_0) \sim (\mu - \mu_0)^{1/2}, \quad e^{\mu_0} = F(Z_0), \quad (3)$$

in the neighborhood of the critical point  $\mu_0$ .

However, if we give up the requirement that the weights  $v_i$  of branching should be positive it is clear that there exists special choices of the  $v_i$  such that not only  $F'(Z_0) = 0$ , but also  $F^{(k)}(Z_0) = 0$ ,  $k = 1, \dots, m-1$ . In this case we obtain

$$Z(\mu) - Z(\mu_0) \sim c(\mu - \mu_0)^{1/m}, \quad e^{\mu_0} = F(Z_0). \quad (4)$$

For  $m > 2$  we say that the ensemble of BP is multicritical [3, 14, 4]. The fractal structure of the BP depends on  $m$ . The Hausdorff dimension of a type  $m$  multicritical BP is

$$d_h(m) = \frac{m}{m-1}. \quad (5)$$

As  $m$  increases the BP gets more and more one-dimensional at the critical point.

The multicritical behavior of the BP can be given a concrete realization as hard dimer models with negative fugacities on BPs with positive weight, very much like the situation for the multicritical matrix models, as pointed out in [15]. We define a BP hard dimer model as

$$Z(\mu, \xi) = \sum_{BP} \prod_i v(i) \prod_l e^{-\mu} \sum_{HD(BP)} \xi^{|HD(BP)|}. \quad (6)$$

Here we will assume all  $v_i$  non-negative and for each BP we sum over all hard dimer configurations. The dimers live on links in the BP, “hard” meaning that the dimers are not allowed to touch. To each dimer we associate a fugacity  $\xi$ . For a given dimer configuration  $HD$  the total weight will thus be  $\xi^{|HD|}$ ,  $|HD|$  denoting the number of dimers in the configuration  $HD$ . To illustrate this let us consider the simplest BP model with positive weight, namely  $v_1, v_3 = 1$ , all other weights equal zero. Thus  $F(Z) = (1 + Z^2)/Z$  and the model indeed has a critical point,  $\mu_0 = \log(2)$ ,  $Z_0 = 1$ . Now allow hard dimers on the links, and let the fugacity of a dimer be  $\xi$ . As shown in Fig. 1 we have now two rooted partition functions, one starting with a link without a dimer ( $Z(\mu, \xi)$ ), the other starting with a link with a dimer ( $\tilde{Z}(\mu, \xi)$ ). The corresponding equations are:

$$e^\mu = \frac{1 + Z^2 + 2Z\tilde{Z}}{Z}, \quad e^\mu = \xi \frac{1 + Z^2}{\tilde{Z}}. \quad (7)$$

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<sup>1</sup>One can obtain non-generic behavior for non-negative weights  $v_i$  if infinite many  $v_i > 0$  and  $F(Z)$ ,  $Z > 0$ , is a monotonic decreasing function in the range where it is defined [17].

For positive fixed values of  $\xi$  one has the standard critical behavior of  $Z$  as a function of  $\mu$ , i.e. a critical point  $\mu_0(\xi)$  such that near that point (3) is valid. However, writing (3) as

$$\mu - \mu_0(\xi) = c_2(\xi) \left( Z(\mu) - Z(\mu_0(\xi)) \right)^2 + c_3(\xi) \left( Z(\mu) - Z(\mu_0(\xi)) \right)^3 + \dots, \quad (8)$$

the coefficient  $c_2$  will decrease as  $\xi$  becomes negative, and eventually one will reach a point  $\xi_c$ , with a critical point  $\mu_c = \mu_0(\xi_c)$  where  $c_2(\xi_c) = 0$ . At the corresponding  $Z_c = Z_0(\mu_c, \xi_c)$  we have

$$\left. \frac{\partial \mu(Z, \xi_c)}{\partial Z} \right|_{Z_c} = \left. \frac{\partial^2 \mu(Z, \xi_c)}{\partial Z^2} \right|_{Z_c} = 0 \quad (9)$$

Clearly this point is a  $m = 3$  multicritical point:

$$\mu - \mu_c \sim c_3 (Z(\mu, \xi_c) - Z_c)^3, \quad \text{i.e.} \quad Z(\mu, \xi_c) - Z_c \sim c(\mu - \mu_c)^{1/3}. \quad (10)$$

One can calculate the critical exponent related to dimers, i.e. the matter system. Following [1, 2] the critical  $\mu_0(\xi)$  has the interpretation as the free energy density of the matter system and the critical exponent  $\sigma$  (corresponding to the magnetization at the critical temperature in the case of a magnetic system) can be defined as

$$\frac{\partial \mu_0(\xi)}{\partial \xi} \sim (\xi - \xi_c)^\sigma \quad (11)$$

where the term on the rhs denotes the leading non-analytic term when  $\xi$  approaches  $\xi_c$ . It is straight forward to show that in addition to (10) we have

$$\xi - \xi_c \sim (Z(\mu_0(\xi), \xi) - Z_c)^2, \quad (12)$$

from which we deduce that  $\sigma = 1/2$ .

The multicritical behavior reported above in terms of dimers is generic for BPs and independent of the particular model. Another explicit BP model, relevant for the theory of causal dynamical triangulations, is one where branching of arbitrary high order, with weight one, is allowed. The graphical equations are shown in Fig. 2 and the equations corresponding to (7) are:

$$e^\mu = \frac{1}{Z} \left( \frac{1}{1-Z} + \frac{\tilde{Z}}{(1-Z)^2} \right), \quad e^\mu \tilde{Z} = \xi \frac{1}{1-Z}. \quad (13)$$

Also this model can be solved explicitly and the critical behavior is as above.

The simplest dimer generalization to higher multicritical points is obtained by considering several kinds of dimers with independent fugacities and allow them

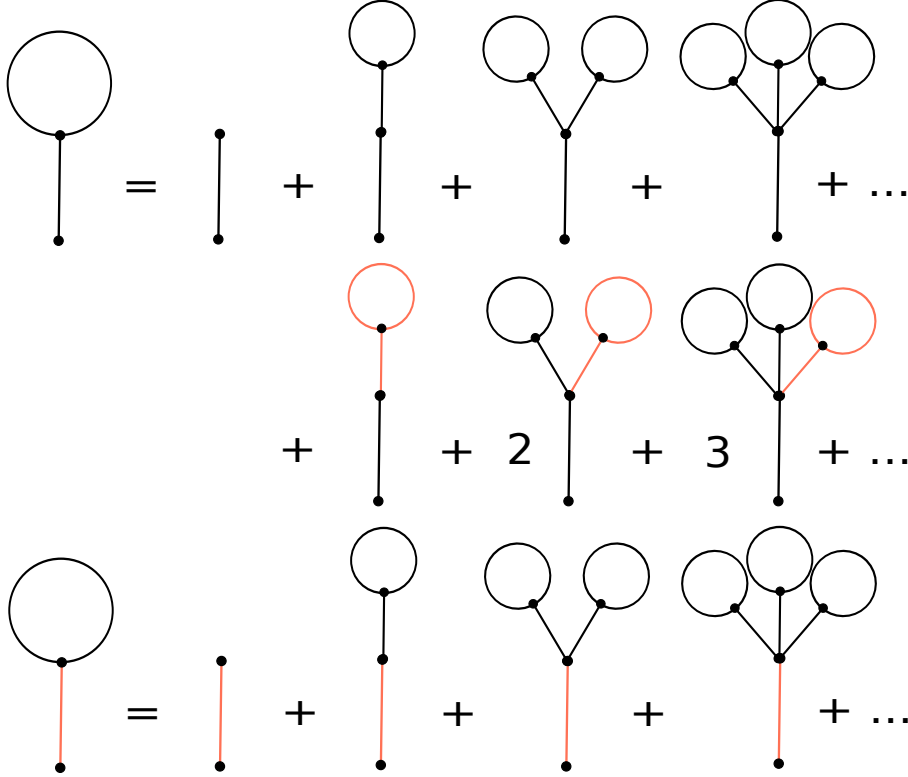


Figure 2: The graphical representation of the equations (13).

to touch the same vertex, but not to be put on the same link. One can then arrange for the fugacities such that one obtains a  $m = 4$  multicritical BP.

We note the following: an Ising model coupled to a generic BP cannot be critical [16]. Although the Hausdorff dimension of the BP is two, part of the inherent structure of the BP is still too one-dimensional to allow for a critical behavior. Thus Ising spins cannot change the fractal properties of an ensemble of BP. However the above calculation shows that coupling hard dimers to BPs can lead to a change of the critical properties of branched polymers. We will comment on the critical properties of the dimers below.

## 2.2 Multicritical CDT

The ordinary two-dimensional CDT theory is defined by the following partition function

$$Z_{CDT}(t) = \sum_{\mathcal{T}} e^{-\mu N(\mathcal{T})}, \quad (14)$$

where the summation is over the set of so-called causal dynamical triangulations  $\mathcal{T}$ , usually defined with two boundaries, separated a link distance  $t$ . We might

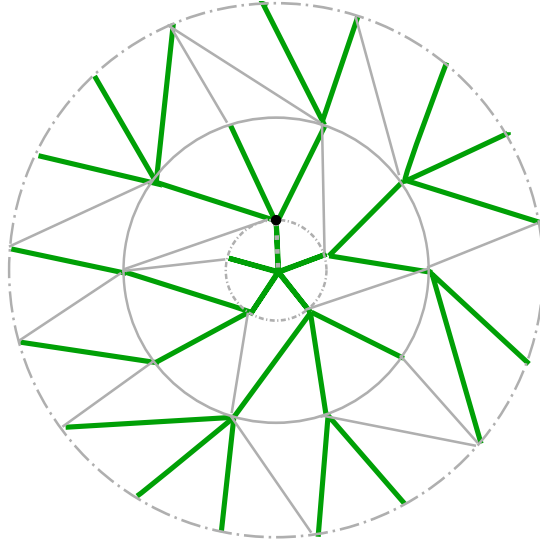


Figure 3: The construction of a tree on a CDT configuration. The inner circle is the entrance boundary. The interior to the inner circle is just an additional construction to obtain bijection between the ensemble of BPs and the CDT ensemble of random surfaces (see [12] for a discussion).

contract the boundaries to contain only a point if convenient. For detail we refer to [10]. There is a bijection between causal triangulations and BPs, as first observed in [12]. Let us mention how this assignment is done. To each vertex in a CDT triangulation, except those at the two boundaries we have a number of links pointing forward in time, a number of links pointing backwards in time and two spatial links. The assignment is now that all links pointing forward from a vertex, except one link which one can take as the link to the left, belong to the BP. For the entrance boundary, which has one marked vertex, a special assignment has to be made. For details we refer to [12] and for an illustration see Fig. 3.

This BP reaches all the vertices of CDT triangulation and to each CDT triangulation there is a distinct BP. Similarly to each BP one can construct a distinct CDT triangulation. The partition functions of the ensembles of BP and CDT are thus identical except for a trivial redefinition of  $\mu$ .

From this identification it is clear that one can add a matter system on the CDT triangulation which changes the critical behavior of the triangulation, namely one can add hard dimers on the BP associated with the CDT triangulation. Adding such dimer systems to the CDT ensemble will change the Hausdorff dimension of the CDT ensemble at the multicritical dimer point, such that a  $m$ -multicritical dimer system,  $m = 3, 4, \dots$  leads to a CDT random surface system with Hausdorff dimension (5). That the Hausdorff dimension of the CDT ensemble is larger than or equal to that of the BP ensemble is clear, since there

are more link-paths connecting two points in a CDT triangulation than in the corresponding the BP-ensemble. If we assume there is a unique fractal structure of the CDT ensemble at all scales at the critical point, it also has to agree with the fractal dimension assigned to the BP. This is because the shortest link distance from the root of a BP to any vertex is essentially the same as in the corresponding CDT triangulation.

These dimer models are of course somewhat artificial viewed from the perspective of the two-dimensional CDT random triangulation. The rule for putting down the dimers (apart from being hard) is that you are not allowed to put them down on spatial links and for the links pointing forward in time from a vertex you are not allowed to put the dimer on the link furthest to the left. Nevertheless one would expect the critical behavior of a fullfledge hard dimer model on the random CDT surfaces to have the same critical behavior. This is what we will show below using the matrix model representation of a generalized CDT model introduced in [13].

### 3 The multicritical CDT matrix models

#### 3.1 Plain CDT

The “plain” CDT model can be obtained as limit of an ordinary matrix model in the following way: assume the following matrix potential

$$V(\phi) = \frac{1}{2}\phi^2 - \lambda\phi - \frac{\lambda}{3}\phi^3, \quad (15)$$

where the linear part of the potential is just for convenience of the scaling limit (see [13] for a discussion), and where  $\phi$  is a  $N \times N$  matrix. We then consider the following partition function:

$$Z(\lambda, g_s) = \int d\phi \, e^{-\frac{N}{g_s} \text{tr} V(\phi)}. \quad (16)$$

Expanding the potential in powers of  $\lambda$  and performing the Gaussian integration results in an ensemble of random surfaces obtained by gluing together triangles (corresponding to the cubic term) and “tadpoles” corresponding to the linear term. The logarithm of  $Z$  represents the sum over the connected surfaces, and taking the large  $N$  limit will single out the surfaces of spherical topology. We will be interested in calculating the disk-amplitude, i.e. planar triangulations with a boundary. The corresponding object is

$$W(x) = \frac{1}{N} \langle \text{tr} \frac{1}{x - \phi} \rangle, \quad (17)$$



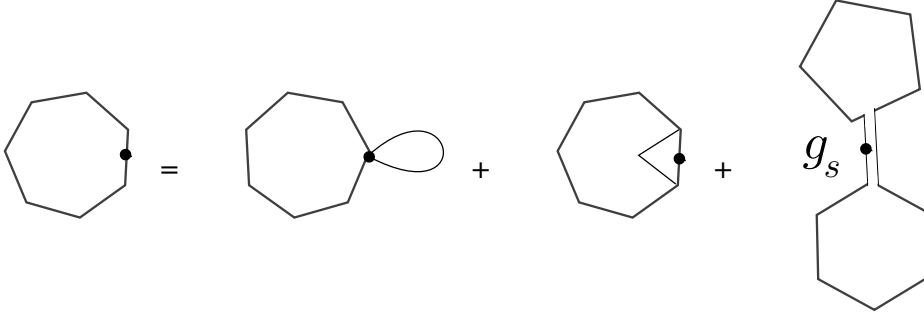


Figure 4: The selfconsistent loop equation to be solved. The coupling constant  $g_s$  associated with the double link monitors the tendency for the creation of baby universes.

where the expectation value is wrt the partition function (16). In the large  $N$  limit the disk amplitude  $W(x)$  satisfies the so-called loop equation

$$g_s W(x)^2 = V'(x)W(x) - Q(x), \quad Q(x) = c_1 x + c_0, \quad (18)$$

where  $V'$  denotes the derivative with respect to  $x$ . The parameter  $x$  is related to a boundary cosmological constant  $\mu_b$  by  $x = e^{\mu_b}$ . The graphic representation of the equation is shown in Fig. 4.

The solution is

$$W(x) = \frac{V'(x) - \sqrt{V'(x)^2 - 4g_s Q(x)}}{2g_s}, \quad (19)$$

and the constants  $c_0$  and  $c_1$  are determined by the requirement that the  $W(x)$  has a single cut on the real axis and that  $W(x) = 1/x + O(1/x^2)$  for  $x \rightarrow \infty$ . Thus  $W(x)$  has the following form

$$W(x) = \frac{V'(x) + \lambda(x - c)\sqrt{(x - b)(x - a)}}{2g_s}. \quad (20)$$

In this formula  $a, b$  and  $c$  are functions of the coupling constants  $\lambda$  and  $g_s$ . The continuum limit where  $W(x)$  can be associated with 2d Euclidean quantum gravity is, for a fixed  $g_s$ , the limit where  $c(\lambda) \rightarrow b(\lambda)$ . In the neighborhood of this point,  $\lambda_c$ , one has the expansion

$$\lambda = \lambda_c(1 - \varepsilon^2)\Lambda, \quad b(\lambda) = b(\lambda_c) - \varepsilon\sqrt{\Lambda}, \quad c(\lambda) = c(\lambda_c) + \frac{1}{2}\varepsilon\sqrt{\Lambda}, \quad (21)$$

where

$$x = x_c + \varepsilon X, \quad x_c = c(\lambda_c) = b(\lambda_c). \quad (22)$$

The interpretation is that  $\Lambda$  is the continuum cosmological constant,  $X$  the continuum boundary cosmological constant. In this limit the term  $V'(x)$  does not scale and one obtains

$$W(x) = NS(x) + \text{const. } \varepsilon^{3/2} W_{cont}(X), \quad (23)$$

$$W_{cont}(X) = (X - \sqrt{\Lambda}/2) \sqrt{X + \sqrt{\Lambda}}, \quad (24)$$

where  $W_{cont}(X)$  agrees with the disk amplitude calculated using quantum Liouville theory and  $NS(x)$  is a non-scaling part, analytic in  $x$ .

It is however possible to take another scaling limit related to CDT when  $g_s \rightarrow 0$ , more specifically we have to scale  $g_s$  as follows

$$g_s = G_s \varepsilon^3. \quad (25)$$

In this limit, which was denoted the “classical” limit, the behavior of the critical couplings to lowest order in  $\varepsilon$  is

$$\lambda_c(g_s) = \frac{1}{2} - \frac{3}{4} G_s^{2/3} \varepsilon^2, \quad x_c(g_s) = 1 + G_s^{1/3} \varepsilon, \quad (26)$$

and the continuum cosmological constant and boundary cosmological constant are defined as

$$\lambda = \lambda_c(g_s) - \varepsilon^2 \Lambda, \quad x = x_c(g_s) + \varepsilon X. \quad (27)$$

Contrary to the situation for a fixed  $g_s$ ,  $W(x)$  itself will now scale and one obtains [13]:

$$W(x) = \varepsilon^{-1} W_{cont}(X), \quad (28)$$

$$W_{cont}(X) = \frac{\Lambda_{\text{cdt}} - \frac{1}{2} X_{\text{cdt}}^2 + (X_{\text{cdt}} - H) \sqrt{(X_{\text{cdt}} + H)^2 - \frac{4G_s}{H}}}{2G_s}, \quad (29)$$

where

$$\Lambda_{\text{cdt}} = \Lambda + \frac{3}{2} G_s^{2/3}, \quad X_{\text{cdt}} = X + G_s^{1/3}, \quad 2\Lambda_{\text{cdt}} H - H^3 = 2G_s. \quad (30)$$

We note for future reference that the critical points

$$(\lambda_*, x_*) := (\lambda_c(g_s = 0), x_c(g_s = 0)) = (1/2, 1) \quad (31)$$

are characterized by

$$V'(\lambda_*, x_*) = V''(\lambda_*, x_*) = 0. \quad (32)$$

In the limit where  $G_s \rightarrow 0$  we obtain precisely the CDT disk function:

$$W_{cont}(X) \rightarrow \frac{1}{X + \sqrt{2\Lambda}} = \frac{1}{X} - \frac{\sqrt{2\Lambda}}{X^2} + \dots \quad (33)$$

$X \rightarrow \infty$  corresponds to contracting the boundary of the disk to a (marked) point, and the leading singularity of the corresponding closed (spherical) surface is thus  $\sqrt{\Lambda}$ , or re-introducing the bare coupling constants,  $(\lambda - \lambda_c(g_s = 0))^{1/2}$ . This critical behavior is precisely the same as the critical behavior of the generic BP behavior, in agreement with the identification of the ensemble of CDT random surfaces with an ensemble of generic BPs mentioned above.

### 3.2 Multicritical “classical” matrix models

We now generalize the construction of a classical limit for “plain” CDT to the multicritical case. We consider the matrix potential

$$V(\phi) = \frac{1}{2}\phi^2 - \lambda\phi - \frac{\lambda}{3}\phi^3 - \frac{\lambda^3\xi}{2}\phi^4. \quad (34)$$

Viewed as a generating potential for random surfaces this matrix potential will glue together triangles and squares. Viewing each square as two triangles, we can think of the squares as part of the triangulation, but with a dimer placed on the diagonal, with a fugacity  $\tilde{\xi} = \lambda\xi$ . In this way the model describes dimers put on random triangulations in a special way, such that there is at most one dimer per triangle. On the graph dual to the triangulation the dimers are precisely hard dimers, and we will call them hard dimers also on the triangulation, even if the rule of putting down the dimers is slightly different from the standard hard dimer rule. Similarly we will denote  $\xi$  the fugacity of the dimers, although it is strictly speaking  $\tilde{\xi}$  which serves as the fugacity.

We are interested in a multicritical behavior of the matrix model (34) and it occurs for negative fugacity  $\xi$ . The disk amplitude is still given by (19) where the polynomial  $Q(x)$  now is of second order with coefficients uniquely fixed by the requirement that we have a single cut on the real line and that  $W(x)$  falls off like  $1/x$  for large  $x$ . The multicritical point  $\lambda_c(g_s), \xi_c(g_s)$  is characterized by the following behavior of  $W(x)$ , generalizing the critical behavior above:

$$W(x) = \frac{V'(x) - 2\lambda_c^2(g_s)\xi_c(g_s)(x - b_c(g_s))^2\sqrt{(x - c_c(g_s))(x - a_c(g_s))}}{2g_s}. \quad (35)$$

For a fixed  $g_s$  the potential  $V'(x)$  does not scale and plays no role in the continuum limit.

We are interested in a “classical” limit of (35) where the potential scales, and that is obtained by the assignment

$$g_s = G_s\varepsilon^4. \quad (36)$$

Note that the scaling in (36) differs from that in (25). This reflects that the multicritical point in the limit  $g_s = 0$  is characterized by the generalization of

(32):

$$V'(\lambda_*, \xi_*, x_*) = V''(\lambda_*, \xi_*, x_*) = V'''(\lambda_*, \xi_*, x_*) = 0. \quad (37)$$

From (34) and (35) it follows that

$$x_* = \frac{1}{\lambda_*} = -\frac{1}{2\xi_*} = \sqrt{3}. \quad (38)$$

Thus a canonical scaling of the boundary cosmological constant around  $x_*$  will lead to a  $V'(\lambda_*, \xi_*, x)$  of order  $\varepsilon^3$  and a behavior compatible with (28) is obtained.

With  $g_s$  scaling like (36) one can calculate  $\lambda_c(g_s), \xi_c(g_s), x_c(g_s)$ :

$$\lambda_c(g_s) = \lambda_* \left( 1 - \frac{\sqrt{5}}{9} G_s^{1/2} \varepsilon^2 - \frac{16\sqrt{5}}{27} G_s^{3/4} \varepsilon^3 \right) + O(\varepsilon^4), \quad (39)$$

$$\xi_c(g_s) = \xi_* \left( 1 - \frac{\sqrt{5}}{9} G_s^{1/2} \varepsilon^2 + \frac{16\sqrt{5}}{27} G_s^{3/4} \varepsilon^3 \right) + O(\varepsilon^4), \quad (40)$$

$$x_c(g_s) = b_c(g_s) = x_* \left( 1 + \frac{2}{3 \cdot 5^{1/4}} G_s^{1/4} \varepsilon \right) + O(\varepsilon^2). \quad (41)$$

The perturbation away from  $\lambda_c(g_s), \xi_c(g_s)$  which leads to a potential  $V'(x, \lambda, \xi)$  of order  $\varepsilon^3$ , assuming the boundary cosmological constant is perturbed as  $x = x_c(g_s) + \varepsilon X$ , can be parametrized as

$$\lambda = \lambda_* + \tilde{\Lambda} \varepsilon^2 - \Lambda \varepsilon^3, \quad \xi = \xi_* - \frac{1}{2} \tilde{\Lambda} \varepsilon^2. \quad (42)$$

As in the ordinary multicritical matrix model situation one finds a two-parameter set of solutions depending on  $\Lambda, \tilde{\Lambda}$ , rather than the solution (29) which only depends on one parameter,  $\Lambda$ . Let us choose a convenient “background”, using the notation from ordinary matrix models [18], which we call the CDT-background, namely  $\tilde{\Lambda} = 0$ . By a redefinition similar to (30):

$$\Lambda_{\text{cdt}} = \Lambda + \frac{32\sqrt{3} \cdot 5^{1/4}}{81} G_s^{3/4}, \quad X_{\text{cdt}} = X + \frac{2}{\sqrt{3} \cdot 5^{1/4}} G_s^{1/4}, \quad (43)$$

we obtain

$$V'(x) = \varepsilon^3 \left( \Lambda_{\text{cdt}} + \frac{1}{9} X_{\text{cdt}}^3 \right). \quad (44)$$

One can now calculate  $W(x)$  in the CDT limit  $G_s \rightarrow 0$  where any creation of baby universes is suppressed:

$$W(x) = \frac{1}{\varepsilon} \frac{1}{X_{\text{cdt}} + \Lambda_{\text{cdt}}^{1/3}}. \quad (45)$$

Repeating the exercise below eq. (33), taking  $X_{\text{cdt}}$  to infinity, we obtain that the critical behavior of the partition function of spherical multicritical CDT surfaces with one marked point is dominated by the term  $(\lambda - \lambda_c(g_s = 0))^{1/3}$ , in agreement with the expectation we have from the multicritical BP model.

Finally, let us note that it follows from (42) that the critical exponent  $\sigma$  of the hard dimers is  $1/2$ , again in complete analogy with the multicritical BP model calculation (6)-(12). The result  $\sigma = 1/2$  is also the result obtained for ordinary multicritical matrix models. In this sense it is not really surprising that we obtain the result. One obtains the CDT multicritical point by following the ordinary multicritical line  $\lambda_c(g_s), \xi_c(g_s)$  from, say,  $g_s = 1$  to the CDT value  $g_s = 0$ . For any fixed value of  $g_s$  we have of course  $\sigma = 1/2$ , and although  $\sigma$  might have changed when taking the particular classical scaling limit  $g_s = \varepsilon^4 G_s$ , this is apparently not what happens.

## 4 Discussion

Only a few riddles are left in 2d Euclidean quantum gravity coupled to matter. One of them is the behavior of the Hausdorff dimension  $d_h$  as a function of the central charge  $c$  of the conformal theory coupled gravity. A formula was derived by Watabiki some years ago [19]

$$d_h = 2 \frac{\sqrt{49 - c} + \sqrt{25 - c}}{\sqrt{25 - c} + \sqrt{1 - c}} \quad (46)$$

Most likely this formula is correct for  $c \leq 0$ . For  $c = 0$  agrees with what is known to be the correct answer [20]. For  $c = -2$  there are very reliable computer simulations which show agreement with the formula [21]. Finally for  $c \rightarrow -\infty$  it gives 2, something one would indeed expect from semiclassical Liouville theory. However, for  $0 < c \leq 1$  the numerical agreement is less conclusive [22], and the possibility that  $d_h = 4$  in this range was pointed out, and the idea has recently been resurrected [23]. For  $c > 1$  the Liouville formulas become complex and expressions like (46) are not valid, but it is believed that there is a universal phase where the world sheet degenerates to BPs.

Surprisingly we have a somewhat similar situation in CDT: from numerical simulations  $d_h$  seems unchanged (and equal 2) when matter with central charge  $0 \leq c \leq 1$  is coupled to the CDT ensemble [24] and recently it was shown that there might be a kind of universal phase for  $c > 1$  [25]. However, to the extent we can really view the hard dimer models as corresponding to conformal field theories, it seems that for  $c < 0$  the matter systems can change fractal structure of the CDT ensemble. Qualitatively the changes are like in the full Euclidean models,  $d_h$  decreases with decreasing  $c$ . In the  $c = 0$  case the CDT model can

be understood as an effective Euclidean model, where baby universes have been integrated out. Whether such an interpretation is possible also when matter is coupled to CDT is presently unknown, but since the multicritical matrix models capture the critical behavior of both CDT and ordinary 2d Euclidean quantum gravity coupled to certain matter systems, depending on how we scale  $g_s$ , we have a chance to answer this question in the context of matrix models.

## Note

While completing this article we were informed by Stefan Zohren that he and Max Atkin have obtained results which are identical to a number of our results. We thank Stefan for informing us of these results prior to publication.

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